Wavelets and Affine Distributions
A Time-Frequency Perspective

Franz Hlawatsch

Institute of Communications and Radio-Frequency Engineering
Vienna University of Technology

INSTITUT FÜR NACHRICHTENTECHNIK UND HOCHFREQUENZTECHNIK

TECHNISCHE UNIVERSITÄT WIEN
VIENNA UNIVERSITY OF TECHNOLOGY

OUTLINE

- The notion of time-frequency analysis
- Linear and quadratic time-frequency analysis
- Short-time Fourier transform and wavelet transform; spectrogram and scalogram
- Constant-bandwidth analysis vs. constant-Q analysis
- The affine class
- Affine time-frequency smoothing
- Hyperbolic time-frequency localization

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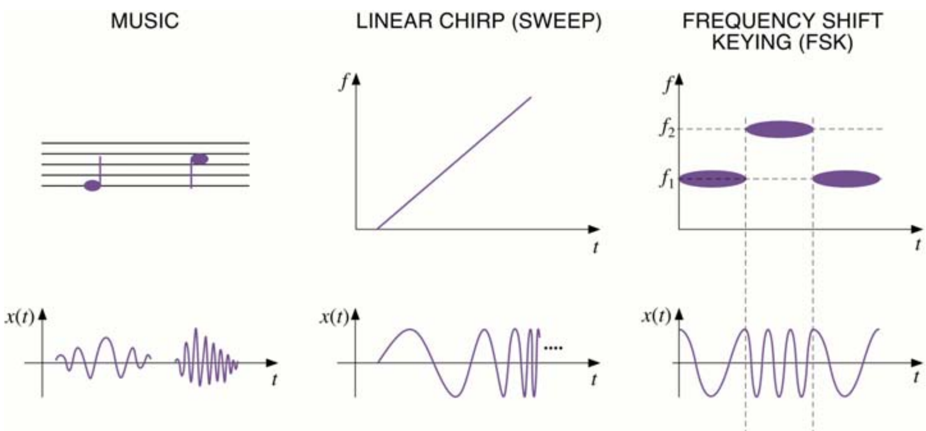
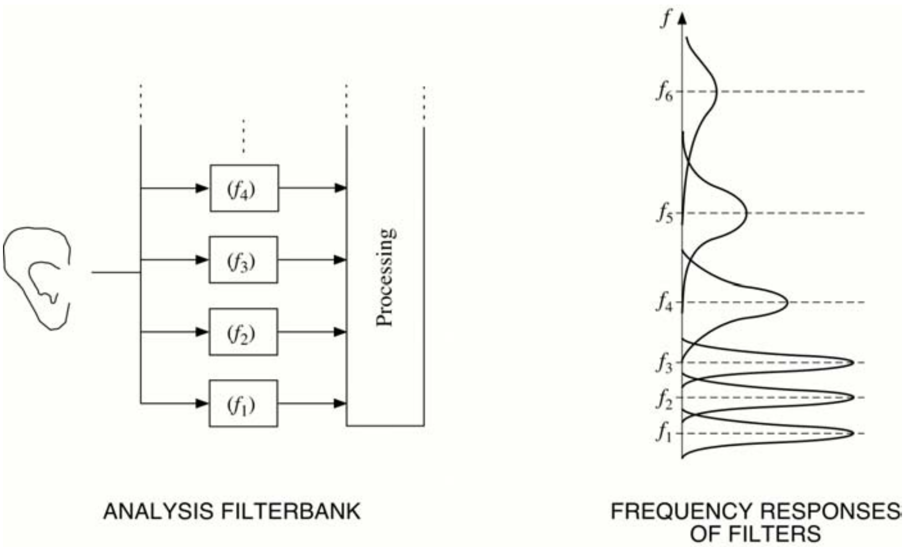
The notion of time-frequency (TF) analysis

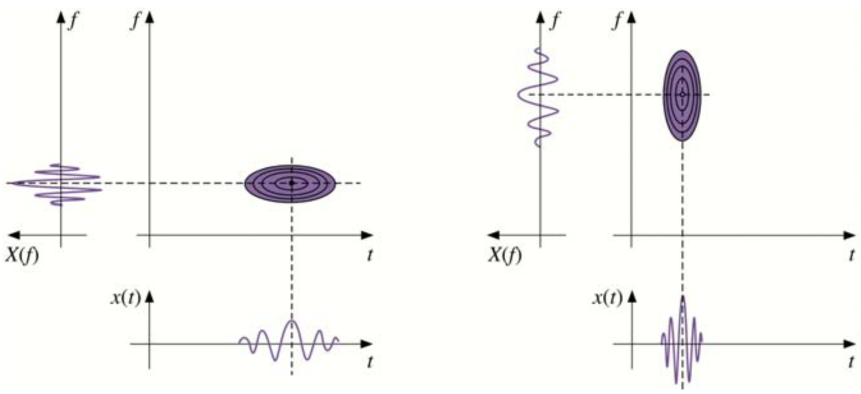


Auditory perception as TF analysis



The TF plane

- Visualize time-frequency location/concentration of signal $x(t)$:



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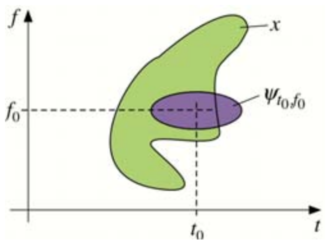
Linear TF analysis

- TF analysis: Measure contribution of TF point (t_0, f_0) to signal $x(t)$
- General approach: Inner product of $x(t)$ with "test signal" or "sounding signal" $\psi_{t_0,f_0}(t)$ located about (t_0, f_0):

$$\mathrm{LTFR}_x(t_0, f_0) := \left\langle x, \psi_{t_0,f_0} \right\rangle = \int_{-\infty}^{\infty} x(t)\, \psi^*_{t_0,f_0}(t)\, dt$$

LTFR = Linear TF Representation



Linear TF synthesis

- TF synthesis (inversion of LTFR): Recover ("synthesize") signal $x(t)$ from $\mathrm{LTFR}_x(t_0, f_0)$
- General approach:

$$x(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{LTFR}_x(t_0, f_0)\, \tilde{\psi}_{t_0,f_0}(t)\, dt_0 df_0$$

$x(t)$ is represented as superposition of TF localized signal components, weighted by "TF coefficient function" $\mathrm{LTFR}_x(t_0, f_0)$

- Problem: How to construct test (analysis) functions $\psi_{t_0,f_0}(t)$ and synthesis functions $\tilde{\psi}_{t_0,f_0}(t)$?

Quadratic TF analysis

- TF analysis: Measure "energy contribution" of TF point (t_0, f_0) to signal $x(t)$
- Simple approach:

$$\mathrm{QTFR}_x(t_0, f_0) := \left|\mathrm{LTFR}_x(t_0, f_0)\right|^2 = \left|\left\langle x, \psi_{t_0,f_0}\right\rangle\right|^2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1) x^*(t_2)\, \psi^*_{t_0,f_0}(t_1) \psi_{t_0,f_0}(t_2)\, dt_1 dt_2$$

QTFR = Quadratic TF Representation

- Want QTFR to distribute signal energy E_x over TF plane:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{QTFR}_x(t, f)\, dt\, df = E_x \qquad \text{"TF energy distribution"}$$

- Problem: How to construct test (analysis) functions $\psi_{t_0,f_0}(t)$?

Construction of analysis/synthesis functions

- Problem: Construct family of analysis functions $\{\psi_{t_0,f_0}(t)\}$ such that $\psi_{t_0,f_0}(t)$ is localized about TF point (t_0, f_0)
- Systematic approach: $\psi_{t_0,f_0}(t)$ derived from "prototype function" $\psi(t)$ via unitary "TF displacement operator" $\mathbf{U}_{t_0,f_0}$:

$$\psi_{t_0,f_0}(t) := \big(\mathbf{U}_{t_0,f_0}\psi\big)(t)$$

- Same for synthesis functions $\{\tilde{\psi}_{t_0,f_0}(t)\}$:

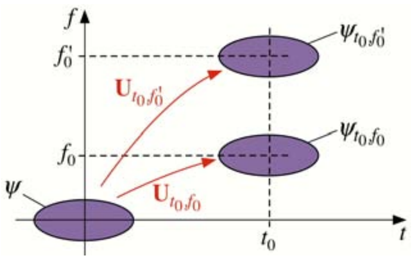
$$\tilde{\psi}_{t_0,f_0}(t) := \big(\mathbf{U}_{t_0,f_0}\tilde{\psi}\big)(t)$$

- Two classical definitions of $\mathbf{U}_{t_0,f_0}$:
  - TF shift
  - TF scaling (compression/dilatation) + time shift

Two classical definitions of operator U

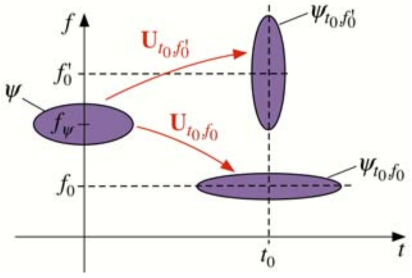
- TF shift:

$$\psi_{t_0,f_0}(t) = \big(\mathbf{U}_{t_0,f_0}\psi\big)(t) = \psi(t-t_0)\, e^{j2\pi f_0 t}$$



- TF scaling + time shift:

$$\psi_{t_0,f_0}(t) = \big(\mathbf{U}_{t_0,f_0}\psi\big)(t) = \sqrt{\frac{f_0}{f_\psi}}\,\psi\Big(\frac{f_0}{f_\psi}(t-t_0)\Big) = \frac{1}{\sqrt{a}}\,\psi\Big(\frac{t-t_0}{a}\Big)\Big|_{a=f_\psi/f_0}$$



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Short-Time Fourier Transform (STFT)

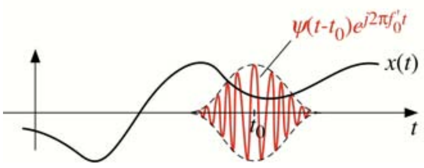
- Recall TF shift:

$$\psi_{t_0,f_0}(t) \;=\; \big(\mathbf{U}_{t_0,f_0}\psi\big)(t) \;=\; \psi(t-t_0)\, e^{j2\pi f_0 t}$$

- $\Rightarrow$ LTFR = STFT:

$$\mathrm{STFT}_x(t_0,f_0) \;=\; \big\langle x, \mathbf{U}_{t_0,f_0}\psi\big\rangle \;=\; \int_{-\infty}^{\infty} x(t)\, \psi^*(t-t_0) e^{-j2\pi f_0 t} dt$$

STFT = FT of local (windowed) segment of $x(t)$:



STFT signal synthesis

- Recall STFT analysis:

$$\mathrm{STFT}_x(t_0, f_0) = \left\langle x, \mathbf{U}_{t_0,f_0}\psi \right\rangle = \int_{-\infty}^{\infty} x(t)\, \psi^*(t-t_0) e^{-j2\pi f_0 t} dt$$

- STFT signal synthesis:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{STFT}_x(t_0, f_0) \left(\mathbf{U}_{t_0,f_0}\tilde{\psi}\right)(t)\, dt_0 df_0 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{STFT}_x(t_0, f_0)\, \tilde{\psi}(t-t_0) e^{j2\pi f_0 t}\, dt_0 df_0 \end{aligned}$$

$x(t)$ is weighted superposition of TF shifted versions of $\tilde{\psi}(t)$

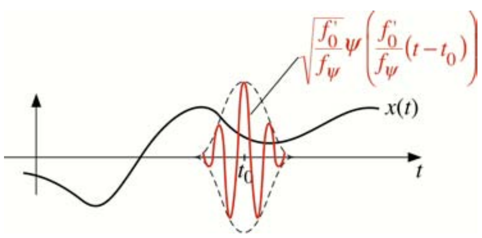
Wavelet Transform (WT)

- Recall TF scaling + time shift:

$$\psi_{t_0,f_0}(t) = \left(\mathbf{U}_{t_0,f_0}\psi\right)(t) = \sqrt{\frac{f_0}{f_\psi}}\, \psi\left(\frac{f_0}{f_\psi}(t-t_0)\right)$$

- $\Rightarrow$ LTFR = WT:

$$\mathrm{WT}_x(t_0, f_0) = \left\langle x, \mathbf{U}_{t_0,f_0}\psi \right\rangle = \int_{-\infty}^{\infty} x(t) \sqrt{\frac{f_0}{f_\psi}}\, \psi^*\left(\frac{f_0}{f_\psi}(t-t_0)\right) dt$$



WT signal synthesis

- Recall WT analysis:

$$\mathrm{WT}_x(t_0,f_0) = \left\langle x, \mathbf{U}_{t_0,f_0}\psi \right\rangle = \int_{-\infty}^{\infty} x(t)\sqrt{\frac{f_0}{f_\psi}}\,\psi^*\!\left(\frac{f_0}{f_\psi}(t-t_0)\right)dt$$

- WT signal synthesis:

$$\begin{aligned} x(t) &= \int_0^{\infty}\int_{-\infty}^{\infty} \mathrm{WT}_x(t_0,f_0)\left(\mathbf{U}_{t_0,f_0}\tilde{\psi}\right)(t)\,dt_0 df_0 \\ &= \int_0^{\infty}\int_{-\infty}^{\infty} \mathrm{WT}_x(t_0,f_0)\sqrt{\frac{f_0}{f_\psi}}\,\tilde{\psi}\!\left(\frac{f_0}{f_\psi}(t-t_0)\right)dt_0 df_0 \end{aligned}$$

$x(t)$ is weighted superposition of TF scaled and time shifted versions of $\tilde{\psi}(t)$

Spectrogram and scalogram

- Recall LTFR $\rightarrow$ QTFR:

$$\mathrm{QTFR}_x(t_0,f_0) = \left|\mathrm{LTFR}_x(t_0,f_0)\right|^2 = \left|\left\langle x, \mathbf{U}_{t_0,f_0}\psi \right\rangle\right|^2$$

- STFT $\rightarrow$ spectrogram:

$$\mathrm{SPEC}_x(t_0,f_0) := \left|\mathrm{STFT}_x(t_0,f_0)\right|^2 = \left|\int_{-\infty}^{\infty} x(t)\,\psi^*(t-t_0)e^{-j2\pi f_0 t}dt\right|^2$$

- WT $\rightarrow$ scalogram:

$$\mathrm{SCAL}_x(t_0,f_0) := \left|\mathrm{WT}_x(t_0,f_0)\right|^2 = \left|\int_{-\infty}^{\infty} x(t)\sqrt{\frac{f_0}{f_\psi}}\,\psi^*\!\left(\frac{f_0}{f_\psi}(t-t_0)\right)dt\right|^2$$

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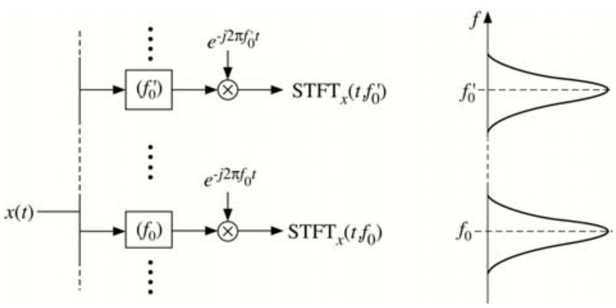
STFT and constant-BW filterbank: analysis

- STFT analysis as convolution:

$$\mathrm{STFT}_x(t_0, f_0) = \int_{-\infty}^{\infty} x(t)\, \psi^*(t-t_0)\, e^{-j2\pi f_0 t}\, dt$$

$$= \left[x(t) * \psi^*(-t)\, e^{j2\pi f_0 t}\right] \cdot e^{-j2\pi f_0 t}\Big|_{t=t_0}$$

- $\Rightarrow$ Filterbank interpretation/implementation:

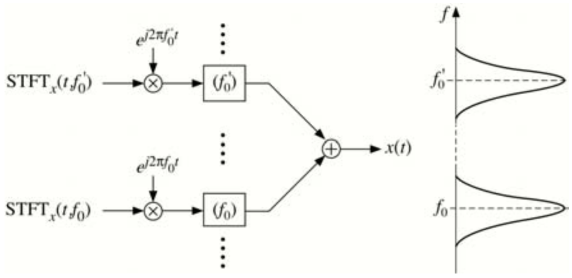


STFT and constant-BW filterbank: synthesis

- STFT synthesis as convolution:

$$x(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{STFT}_x(t_0, f_0)\, \tilde{\psi}(t-t_0) e^{j2\pi f_0 t}\, dt_0 df_0$$
$$= \int_{-\infty}^{\infty} \left[\mathrm{STFT}_x(t_0, f_0) e^{j2\pi f_0 t_0} * \tilde{\psi}(t_0) e^{j2\pi f_0 t_0}\right]_{t_0=t} df_0$$

- $\Rightarrow$ Filterbank interpretation/implementation:

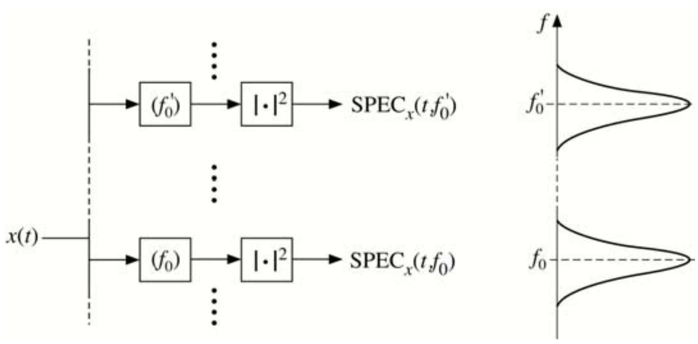


Spectrogram analysis as constant-BW filterbank

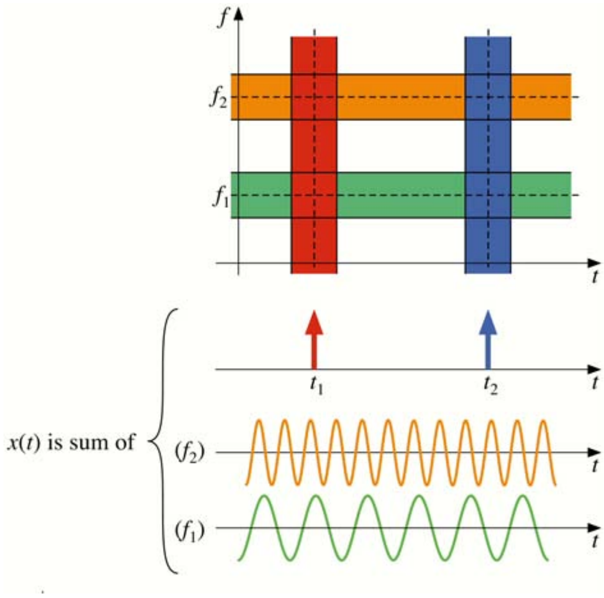
- Spectrogram analysis as convolution:

$$\mathrm{SPEC}_x(t_0, f_0) = \left|\mathrm{STFT}_x(t_0, f_0)\right|^2 = \left|\left[x(t) * \psi^*(-t) e^{j2\pi f_0 t}\right]_{t=t_0}\right|^2$$

- $\Rightarrow$ Filterbank interpretation/implementation:



STFT / spectrogram: example



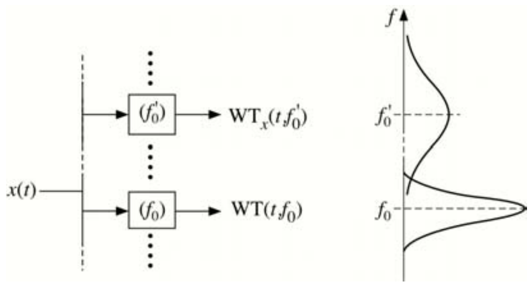
WT and constant-Q filterbank: analysis

- WT analysis as convolution:

$$\mathrm{WT}_x(t_0, f_0) = \int_{-\infty}^{\infty} x(t) \sqrt{\frac{f_0}{f_\psi}}\, \psi^*\!\left(\frac{f_0}{f_\psi}(t-t_0)\right) dt$$

$$= \left[x(t) * \sqrt{\frac{f_0}{f_\psi}}\, \psi^*\!\left(-\frac{f_0}{f_\psi}\, t\right) \right]_{t=t_0}$$

- $\Rightarrow$ Filterbank interpretation/implementation:

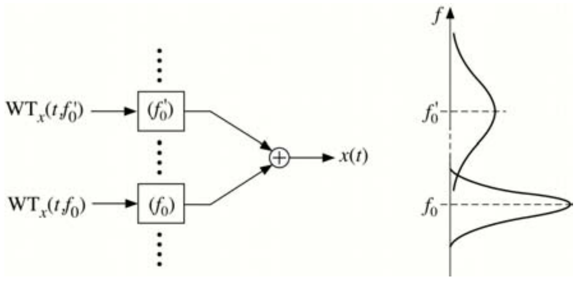


WT and constant-Q filterbank: synthesis

- WT synthesis as convolution:

$$x(t) = \int_0^\infty \int_{-\infty}^\infty \mathrm{WT}_x(t_0, f_0) \sqrt{\frac{f_0}{f_\psi}}\, \tilde{\psi}\Big(\frac{f_0}{f_\psi}(t - t_0)\Big)\, dt_0 df_0$$
$$= \int_0^\infty \Big[\mathrm{WT}_x(t_0, f_0) * \sqrt{\frac{f_0}{f_\psi}}\, \tilde{\psi}\Big(\frac{f_0}{f_\psi} t_0\Big)\Big]_{t_0 = t} df_0$$

- $\Rightarrow$ Filterbank interpretation/implementation:

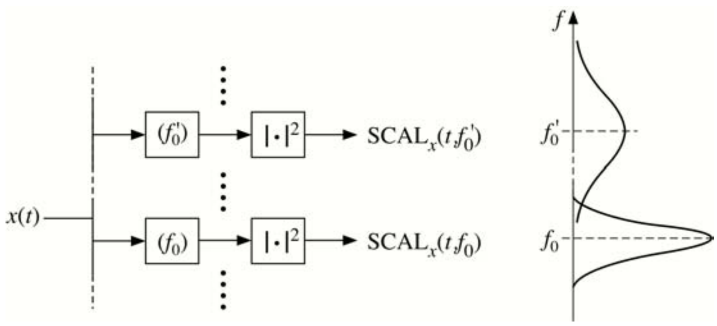


Scalogram analysis as constant-Q filterbank

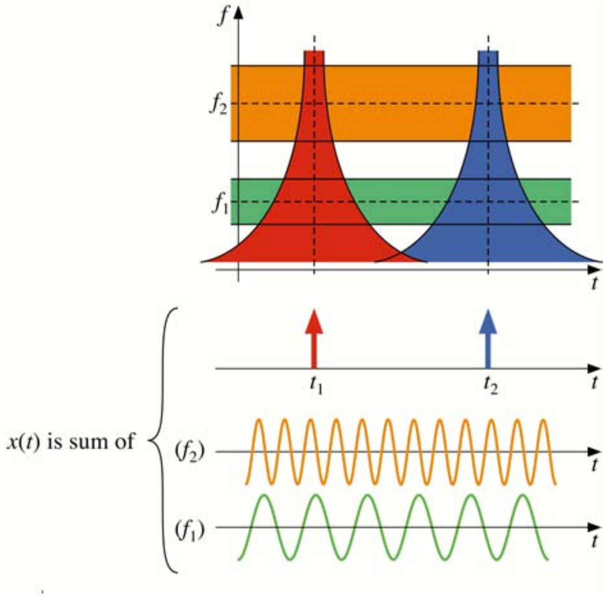
- Scalogram analysis as convolution:

$$\mathrm{SCAL}_x(t_0, f_0) = \big|\mathrm{WT}_x(t_0, f_0)\big|^2 = \left| \Big[x(t) * \sqrt{\frac{f_0}{f_\psi}}\, \psi^*\Big(-\frac{f_0}{f_\psi} t\Big)\Big]_{t=t_0}\right|^2$$

- $\Rightarrow$ Filterbank interpretation/implementation:

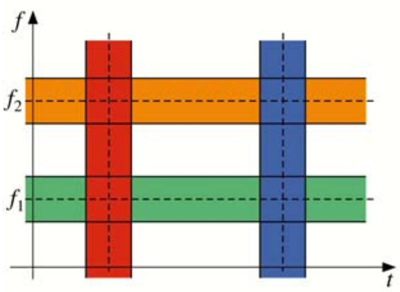


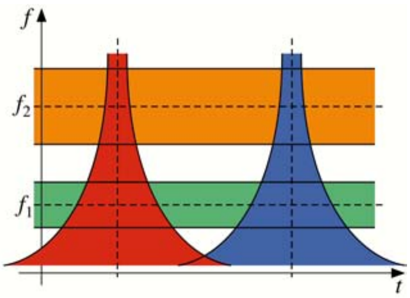
WT / scalogram: example



STFT / spectrogram vs. WT / scalogram

STFT / spectrogram **WT / scalogram**





Good-bye and hello

- Good-bye to:
  - STFT
  - spectrogram
  - constant-BW analysis
- Hello to:
  - affine class of QTFRs
  - Wigner distribution and Bertrand distribution
  - hyperbolic TF localization

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Axiomatic (covariance-based) definition of WT

- Generic LTFR expression:

$$\mathrm{LTFR}_x(t,f) \;=\; \int_{-\infty}^{\infty} x(t')\, K(t';t,f)\, dt'$$

- Covariance of LTFR to TF scalings + time shifts:

$$y(t) \;=\; \frac{1}{\sqrt{a}}\, x\Big(\frac{t-\tau}{a}\Big) \quad\leftrightarrow\quad Y(f) \;=\; \sqrt{a}\, X(af)\, e^{-j2\pi\tau f}$$

$$\Rightarrow\; \mathrm{LTFR}_y(t,f) \;=\; \mathrm{LTFR}_x\Big(\frac{t-\tau}{a}, af\Big)$$

- Can show that covariant LTFRs are given by WT

$$\mathrm{WT}_x(t,f) \;=\; \int_{-\infty}^{\infty} x(t')\, \sqrt{\frac{f}{f_\psi}}\, \psi^*\Big(\frac{f}{f_\psi}(t'-t)\Big)\, dt' \;=\; \sqrt{f}\int_{-\infty}^{\infty} x(t')\, \phi\big(f(t'-t)\big)\, dt'$$

Axiomatic (covariance-based) definition of the affine class of QTFRs

- Generic QTFR expression:

$$\mathrm{QTFR}_x(t,f) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)x^*(t_2)\, K(t_1,t_2;t,f)\, dt_1 dt_2$$

- Covariance of QTFR to TF scalings + time shifts:

$$y(t) \;=\; \frac{1}{\sqrt{a}}\, x\Big(\frac{t-\tau}{a}\Big) \quad\leftrightarrow\quad Y(f) \;=\; \sqrt{a}\, X(af)\, e^{-j2\pi\tau f}$$

$$\Rightarrow\; \mathrm{QTFR}_y(t,f) \;=\; \mathrm{QTFR}_x\Big(\frac{t-\tau}{a}, af\Big)$$

- Can show that covariant QTFRs are given by

$$\mathrm{AC}_x(t,f) \;=\; f\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)x^*(t_2)\, \phi\big(f(t_1-t), f(t_2-t)\big)\, dt_1 dt_2$$

AC = Affine Class

The affine class of QTFRs

- Affine class of QTFRs:

$$\mathrm{AC}_x(t,f) = f \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) x^*(t_2)\, \phi\big(f(t_1-t), f(t_2-t)\big)\, dt_1 dt_2$$

- 2-D "kernel" $\phi(\alpha_1, \alpha_2)$ specifies QTFR of the AC
- Scalogram is a member of the AC; its kernel is *separable*:

$$\phi(\alpha_1,\alpha_2) = \frac{1}{f_\psi}\, \psi^*\Big(\frac{\alpha_1}{f_\psi}\Big)\, \psi\Big(\frac{\alpha_2}{f_\psi}\Big)$$

- Expression of AC QTFRs in terms of signal's FT:

$$\mathrm{AC}_x(t,f) = \frac{1}{f} \int_0^{\infty} \int_0^{\infty} X(f_1) X^*(f_2)\, \Phi\Big(\frac{f_1}{f}, \frac{f_2}{f}\Big)\, e^{j2\pi(f_1-f_2)t} df_1 df_2$$

Affine class and affine group

- TF scaling + time shift:

$$\big(\mathbf{U}_{a,\tau}\, x\big)(t) = \frac{1}{\sqrt{a}}\, x\Big(\frac{t-\tau}{a}\Big) = \sqrt{\beta}\, x(\beta t + \gamma) =: \big(\tilde{\mathbf{U}}_{\beta,\gamma}\, x\big)(t)$$

- Affine time transformation $t \to \beta t + \gamma$ ("clock change")
- Composition of clock changes is another clock change:

$$\tilde{\mathbf{U}}_{\beta_2,\gamma_2} \tilde{\mathbf{U}}_{\beta_1,\gamma_1} = \tilde{\mathbf{U}}_{\beta_1\beta_2,\, \gamma_1+\beta_1\gamma_2}$$

- $\Rightarrow$ $\tilde{\mathbf{U}}_{\beta,\gamma}$ is unitary representation of the affine group:
  - Set: $(\beta,\gamma) \in \mathbb{R}^+ \times \mathbb{R}$
  - Group operation: $(\beta_1,\gamma_1) \circ (\beta_2,\gamma_2) = (\beta_1\beta_2,\, \gamma_1 + \beta_1\gamma_2)$
  - Neutral element: $(\beta_0,\gamma_0) = (1,0)$

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The Wigner-Ville Distribution (WVD)

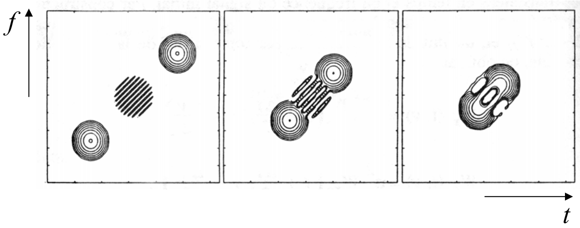
- Prominent member of the AC: the WVD

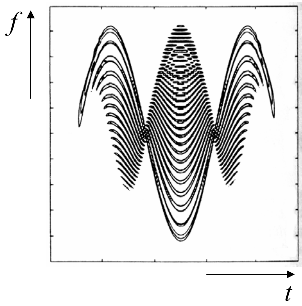
$$\mathrm{WVD}_x(t,f) = \int_{-\infty}^{\infty} x\Big(t+\frac{\tau}{2}\Big)\, x^*\Big(t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau$$

- Properties of the WVD:
  - Covariant to TF scaling and time shift (of course)
  - Covariant to frequency shift ⇒ *not constant-Q*
  - Real for any (real or complex) signal $x(t)$
  - Marginal properties: e.g., $\int_{-\infty}^{\infty} \mathrm{WVD}_x(t,f)\, dt \;=\; |X(f)|^2$
  - Localization properties: e.g., $\mathrm{WVD}_x(t,f) = \delta(f-f_0)$ for $x(t) = e^{j2\pi f_0 t}$
  - Many more…

Interference terms in the WVD

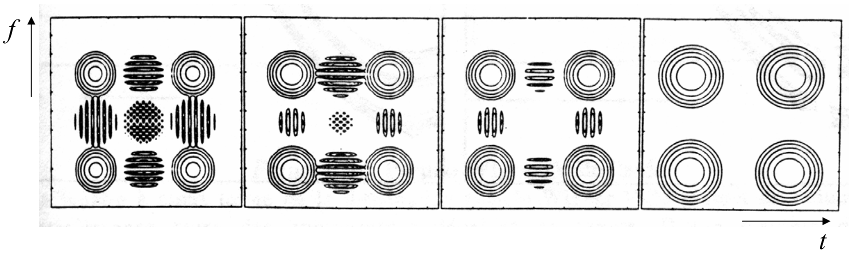
$$\mathrm{WVD}_{x_1+x_2}(t,f) = \mathrm{WVD}_{x_1}(t,f) + \mathrm{WVD}_{x_2}(t,f) + \underbrace{2\,\mathrm{Re}\left\{\mathrm{WVD}_{x_1,x_2}(t,f)\right\}}_{\text{Interference/cross term}}$$





Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

Constant-BW smoothing of the WVD



Smaller/less interference terms

Poorer TF resolution

Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

AC expression in terms of WVD

- Any QTFR of the AC can be expressed in terms of the WVD:

$$\mathrm{AC}_x(t,f) = \int_0^\infty \int_{-\infty}^\infty \mathrm{WVD}_x(t',f')\,\sigma\!\left(f(t'-t), \frac{f'}{f}\right) dt'\,df',$$

  where $\sigma(\alpha,\beta)$ is related to $\phi(\alpha_1,\alpha_2)$ and $\Phi(\beta_1,\beta_2)$ by FTs

- If $\sigma(\alpha,\beta)$ is a *smooth* function, then $\mathrm{AC}_x(t,f)$ is a smoothed version of $\mathrm{WVD}_x(t,f)$

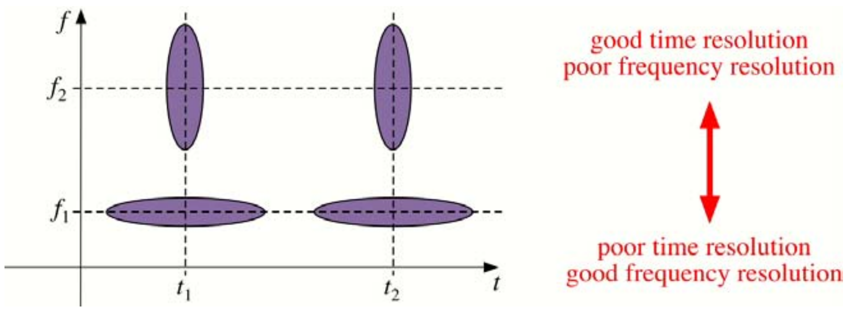
- Smoothing causes…
  - smaller/less interference terms
  - poorer TF resolution

Affine (constant-Q) smoothing, different from constant-BW smoothing shown on previous slide!

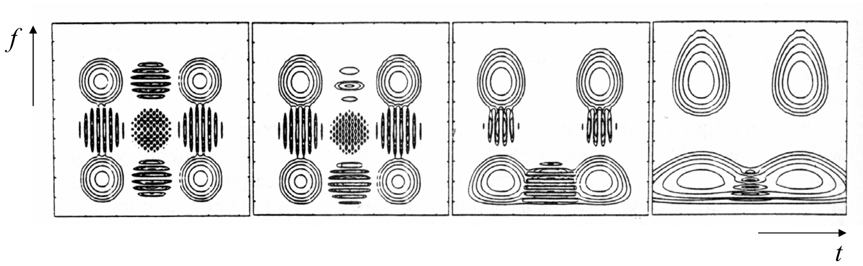
Affine (constant-Q) smoothing of the WVD

- Recall: $\mathrm{AC}_x(t,f) = \int_0^\infty \int_{-\infty}^\infty \mathrm{WVD}_x(t',f')\,\underbrace{\sigma\!\left(f(t'-t), \frac{f'}{f}\right)}_{\text{Smoothing function}} dt'\,df'$

- Smoothing function $\sigma\!\left(f(t'-t), \frac{f'}{f}\right)$ at various TF positions:



Affine smoothing: example



Smaller/less interference terms

Poorer TF resolution

Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

Scalogram as smoothed WVD

- Recall scalogram:

$$\mathrm{SCAL}_x(t,f) := \left|\mathrm{WT}_x(t,f)\right|^2 = \left|\int_{-\infty}^{\infty} x(t')\sqrt{\frac{f}{f_\psi}}\,\psi^*\!\left(\frac{f}{f_\psi}(t'-t)\right)dt'\right|^2$$

- Expression of scalogram as smoothed WVD:

$$\mathrm{SCAL}_x(t,f) = \int_0^\infty \int_{-\infty}^{\infty} \mathrm{WVD}_x(t',f')\, \underbrace{\mathrm{WVD}_\psi\!\left(\frac{f}{f_\psi}(t'-t), \frac{f'}{f/f_\psi}\right)}_{\text{Smoothing function is WVD of wavelet:}} dt'df'$$

Smoothing function
is WVD of wavelet:

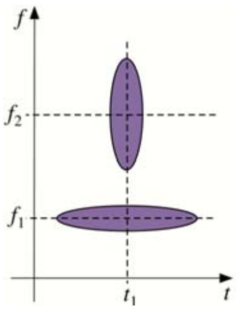
$$\sigma(\alpha,\beta) = \mathrm{WVD}_\psi\!\left(\frac{\alpha}{f_\psi}, f_\psi\beta\right)$$

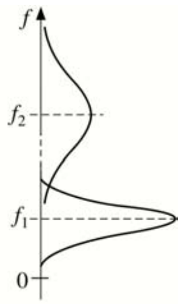
Affine WVD smoothing and constant-Q analysis

- Scalogram as smoothed WVD:

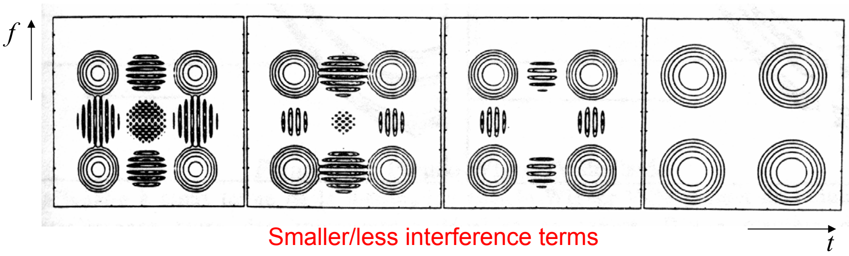
$$\mathrm{SCAL}_x(t,f) = \int_0^{\infty}\int_{-\infty}^{\infty} \mathrm{WVD}_x(t',f')\ \mathrm{WVD}_\psi\left(\frac{f}{f_\psi}(t'-t), \frac{f'}{f/f_\psi}\right) dt'df'$$

$$= \left| \int_{-\infty}^{\infty} x(t') \sqrt{\frac{f}{f_\psi}}\, \psi^*\left(\frac{f}{f_\psi}(t'-t)\right) dt' \right|^2$$



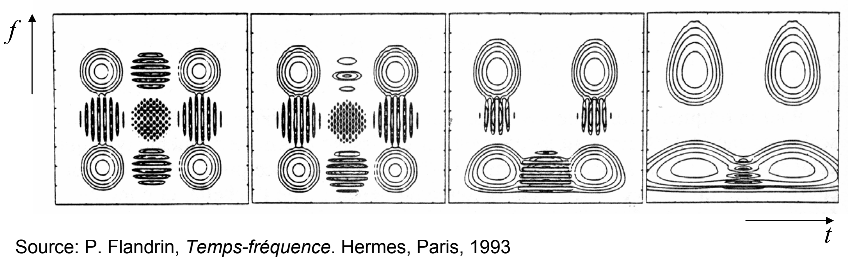


Constant-BW vs. affine (constant-Q) smoothing



Smaller/less interference terms

Poorer TF resolution



Source: P. Flandrin, *Temps-fréquence*. Hermes, Paris, 1993

OUTLINE

- The notion of time-frequency analysis
- Linear and quadratic time-frequency analysis
- Short-time Fourier transform and wavelet transform; spectrogram and scalogram
- Constant-bandwidth analysis vs. constant-Q analysis
- The affine class
- Affine time-frequency smoothing
- ***Hyperbolic time-frequency localization***

Doppler-tolerant signals

- TF scaling / Doppler effect:

$$(\mathbf{C}_a\, x)(t) = \frac{1}{\sqrt{a}}\, x\!\left(\frac{t}{a}\right) \quad \leftrightarrow \quad \sqrt{a}\, X(af)$$

- "Doppler-tolerant" signal = eigenfunction of $\mathbf{C}_a$:

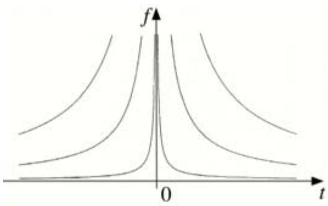
$$(\mathbf{C}_a\, x)(t) = \lambda_a\, x(t)$$

- Solution: "hyperbolic impulse"

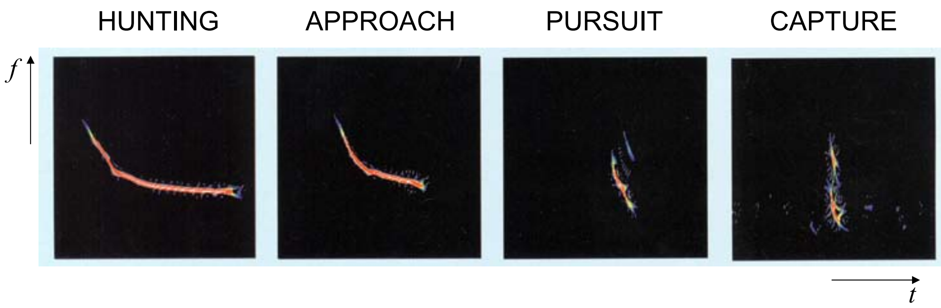
$$X(f) = H_c(f) = \frac{1}{\sqrt{f}}\, e^{-j2\pi c \ln(f/f_r)}, \quad f > 0\,,\ c \in \mathbb{R}$$

- Group delay:

$$\tau(f) = -\frac{1}{2\pi}\frac{d}{df}\arg\{H_c(f)\} = \underbrace{\frac{c}{f}}_{\text{Hyperbola in the TF plane}}$$



Example: Bat sonar signals

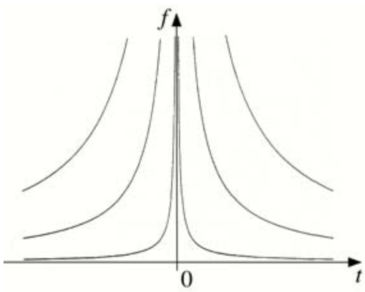


Source: P. Flandrin

Hyperbolic TF localization

- Want AC QTFR to satisfy hyperbolic TF localization property:

$$X(f) = \frac{1}{\sqrt{f}} e^{-j2\pi c \ln(f/f_r)} \quad \Rightarrow \quad \mathrm{AC}_x(t,f) = \frac{1}{f}\,\delta\!\left(t - \frac{c}{f}\right)$$



- *Not* satisfied by WVD !

The Bertrand P_0 distribution

- The hyperbolic TF localization property is satisfied by the (unitary) Bertrand P_0 distribution

$$\mathrm{BER}_x(t,f) = f \int_{-\infty}^{\infty} X\big(f\,\lambda(u)\big)\, X^*\big(f\,\lambda(-u)\big)\, e^{j2\pi f t u} \mu(u)\, du\,, \quad f>0$$

with

$$\lambda(u) = \frac{e^{u/2}\, u/2}{\sinh(u/2)}\,, \qquad \mu(u) = \frac{u/2}{\sinh(u/2)}$$

- The Bertrand P_0 distribution is a central member of the AC. It satisfies several important properties (besides the hyperbolic TF localization property).

Bertrand P_0 distribution as generator of the AC

- Any QTFR of the AC can be expressed in terms of the Bertrand P_0 distribution:

$$\mathrm{AC}_x(t,f) = \int_0^{\infty}\int_{-\infty}^{\infty} \mathrm{BER}_x(t',f')\, \tilde{\sigma}\Big(f\,(t'-t), \frac{f'}{f}\Big)\, dt'df',$$

where $\tilde{\sigma}(\alpha,\beta)$ is related to $\sigma(\alpha,\beta)$

- Special case: scalogram

$$\mathrm{SCAL}_x(t,f) = \int_0^{\infty}\int_{-\infty}^{\infty} \mathrm{BER}_x(t',f')\, \underbrace{\mathrm{BER}_\psi\Big(\frac{f}{f_\psi}(t'-t), \frac{f'}{f/f_\psi}\Big)}_{\text{Smoothing function is BER of wavelet:}}\, dt'df'$$

Smoothing function
is BER of wavelet:

$$\tilde{\sigma}(\alpha,\beta) = \mathrm{BER}_\psi\Big(\frac{\alpha}{f_\psi}, f_\psi \beta\Big)$$

Mellin transform and hyperbolic marginals

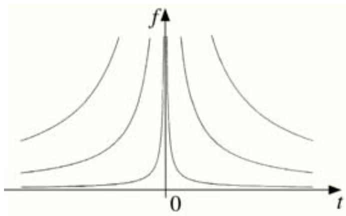
- Recall *hyperbolic impulse* $H_c(f) = \frac{1}{\sqrt{f}} e^{-j2\pi c \ln(f/f_r)}$
- Mellin transform:

$$M_x(c) = \langle X, H_c \rangle = \int_0^\infty X(f)\, e^{j2\pi c \ln(f/f_r)} \frac{df}{\sqrt{f}}$$

- Hyperbolic marginal property:

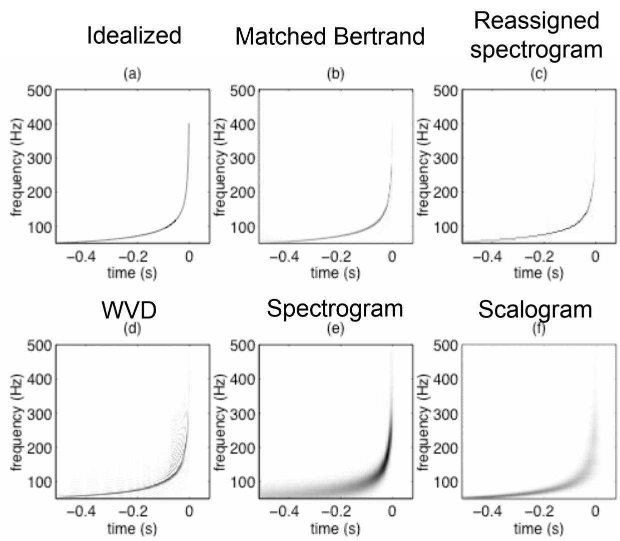
$$\underbrace{\int_0^\infty \mathrm{AC}_x\left(\frac{c}{f}, f\right) \frac{df}{f}}_{} = |M_x(c)|^2$$

Integrate $\mathrm{AC}_x(t,f)$ over TF hyperbola $t=c/f$



- *Not* satisfied by WVD… but satisfied by Bertrand P_0 distri-bution !

Application: TF analysis of gravitational wave



Source: E. Chassande-Mottin and P. Flandrin, On the time-frequency detection of chirps. *Appl. Comp. Harm. Anal.*, 6(9): 252-281, 1999.

Conclusion

- Linear and quadratic TF analysis
- Short-time Fourier transform and spectrogram
- Wavelet transform and scalogram
- Filterbank interpretation: constant-BW analysis versus constant-Q analysis
- Scaling/shift covariance and affine class of QTFRs
- Wigner-Ville distribution and affine smoothing
- Doppler tolerance and hyperbolic impulses
- Hyperbolic TF localization and Bertrand P_0 distribution
- Mellin transform and hyperbolic marginal property

WARNING

YOU ARE LEAVING THE TIME-FREQUENCY PLANE